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Ms. Taylor's Choice: An Instructional Minicase on Selecting Examples to Support Student Understanding of the Associative and Commutative Properties

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Abstract

There is a broad consensus that beginning teachers of mathematics need a strong foundation in mathematical knowledge for teaching (MKT), defined as the mathematical knowledge required to recognize, understand, and respond to the mathematical work of teaching one must engage in. One recurrent challenge in teacher education is how to provide support for preservice teachers (PSTs) to acquire such competencies. Recent trends toward practice-based teacher education support the idea of engaging novice teachers in activities that are purposefully constrained to a core teaching practice. "Ms. Taylor's Choice" is an abbreviated instructional case (i.e., a minicase) based on an assessment scenario in which PSTs must select an example that will focus students' attention on how the associative and commutative properties can be used to evaluate expressions. PSTs are asked to take into account the instructional goals, the motivations of students, and the difficulty of the mathematics in choosing from among these examples as a way of further developing their own MKT.

Keywords mathematics education, associative property, selecting examples, mathematical knowledge for teaching, teacher preparation, commutative property

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We plan to increase the number of minicases in the coming years and to make further improvements based on feedback from those using the materials. If you would like to make suggestions, please contact Heather Howell at hhowell@ets.org.

On the following pages, we present the fruits of a line of work that has spanned multiple projects over multiple years and reflects the contributions of a number of individuals at different points in time. The rationale for the minicase's development is, in essence, quite simple. Much of recent scholarship on teachers' mathematical knowledge for teaching (MKT) has focused on the assessment of MKT via practice-based questions. Practice-based questions generally include a short introductory scenario whose features are critical in solving the task. These scenarios are not simply window dressing for the task, but rather, along with the specified mathematical content, they codefine what is measured (Phelps & Howell, 2016). As such, these tasks can be understood to constitute abbreviated representations of teaching practice (Lai et al., 2013).

Because there has been intense interest in the field in assessing MKT, sample assessment tasks currently make up much of the field's description of specific MKT. Since such assessments became available, we have been approached by several teacher educators interested in integrating MKT assessment items into the curricular content of their mathematics and mathematics methods courses, not by using them as assessments, but by using them as exemplar instructional cases (see Lai & Howell, 2014, for example tasks). However, a number of obstacles to this kind of use have been noted, leading teacher educators to request publicly available full sets of materials that are aligned to instructional goals. Our goal in developing the minicases was to take on some of these challenges by developing a set of support materials designed to aid teacher educators in making use of the items as a curricular resource and at the same time to illustrate one type of support that could be developed more broadly out of such items.

The development team consists of researchers in mathematics and mathematics education, as well as current teacher educators. This work began initially as part of a 2011 project at Educational Testing Service (ETS) intended to investigate the design features of MKT items in the hope of identifying relationships between structural features of the items and ways they performed well in measuring MKT. This project used released items from the Measures of Effective Teaching project (https://k12education.gatesfoundation.org/blog/measures-of-effective-teaching-met-project) and, for each item, created an analytic memo, one purpose of which was to document the reasoning a test taker might use in responding, clearly identifying in each case not just why the intended answer was best but also the logical basis on which each of the competing answer choices could be discarded. The other purpose was to map that reasoning to types of specialized, common, and pedagogical knowledge as described in the Ball et al.

(2008) theory of MKT. Over the subsequent year, the team worked to refine these documents and tailor them to the possibility of serving multiple audiences, including item writers, researchers, teacher educators, and test takers themselves. We used this documentation in a validity study (Howell, Phelps, et al., 2013) and disseminated it at a number of conferences (Howell et al., 2017; Howell & Mikeska, 2016; Howell et al., 2016; Howell, Weren, & Ruiz Diaz, 2013; Lai & Howell, 2014; Phelps et al., 2013), where we received critical feedback but also an enthusiastic reception from teacher educators eager to see and use more of these items. In 2013, a separate National Science Foundation (NSF) funded project[1] (https://www.nsf.gov/awardsearch/showAward?AWD_ID=1445630) created a set of secondary-level MKT items with accompanying documentation and collected similar validity evidence (Lai & Howell, 2016). The project furthered our dissemination goals by creating a Google group in which the items and documentation are housed and available to interested parties.

With a critical mass of systematic assessment documentation at hand, we decided to further develop this material into a set of MKT minicases, documents designed to be used directly by teacher educators in supporting preservice teachers' development of MKT. We chose the name "minicase" to distinguish these materials from "instructional cases" (L. S. Shulman, 1986; Stake, 1987) because they differ from each other in structure and in degree of specificity (J. H. Shulman, 1992). The minicases are shorter than many cases used in professional preparation and are not structured to reveal additional information beyond the initial scenario. The minicases also target very specific knowledge about teaching and learning and are less open to interpretation than most instructional cases. In 2016 and 2017, ETS funded the development of four minicases (two at elementary level and two at secondary level) based on teacher educator input. In 2018, we solicited reviews of the materials from four researchers in the fields of mathematics and mathematics education, as well as from six practicing teacher educators. The feedback from these reviews was then used to revise the set of four minicases to improve mathematical accuracy and comprehensiveness, as well as usability.

Background

There is a broad consensus that beginning teachers of mathematics need a strong foundation in mathematical knowledge for teaching (MKT), defined as the mathematical knowledge required to recognize, understand, and respond to the mathematical work of teaching one must engage in (Ball et al., 2008). Standards call out, for example, competencies for beginning teachers such as "possessing robust knowledge of mathematical and statistical knowledge and concepts," "expanding and deepening [preservice teachers'] knowledge of students as learners of mathematics," and engaging in "effective and equitable mathematics teaching practice" (Association of Mathematics Teacher Educators, 2017, p. 6). One recurrent challenge in teacher education is finding ways to provide support for preservice teachers (PSTs) to acquire such competencies. Recent trends toward practice-based teacher education support the idea of engaging novice teachers in activities that are purposefully constrained to a core teaching practice (Ball & Forzani, 2009). The MKT minicases we have developed represent one such example.

Research on using cases for subject-specific teacher learning goes as far back as the 1990s (Sykes & Bird, 1992). In mathematics and teacher education, cases can also provide a common language, explicit expectations of high-quality mathematics teaching, information about K–12 student development and common misunderstandings, and a means to interact with challenging content (Barnett, 1991).

Each minicase includes a situated task of teaching practice originally developed as part of teacher assessment efforts. Our guiding hypothesis is that these assessment scenarios, along with the accompanying documents that make up the minicases, form a set of resources for teacher educators. These resources are designed to support instructional goals, including developing PSTs' understanding of K–12 student and higher-level mathematics, developing PSTs' orientations toward K–12 students and student work, helping PSTs understand what makes up the professional work of teaching mathematics, and providing them opportunities to engage in the cognitive work associated with addressing the given task.

Because each situated task was originally designed for assessment purposes and crafted to have a single best answer, the resulting minicases require users to take a stand with respect to the presented problem. These cases, unlike instructional cases that are more open-ended, invite response and disagreement in a way that can support rich but focused discussion. Our intention is to support teacher educators who are teaching math methods courses or math content courses for PSTs by providing a set of materials that can be used flexibly and adapted as appropriate.

Instructional Task: The Taylor Item

A lesson in Ms. Taylor's textbook states the associative and commutative properties of addition. To motivate the students to learn the properties, she tells her students that the properties can often be used to simplify the evaluation of expressions.

She wants to give her students an example that will focus their attention on how these properties can be useful in evaluating expressions. Of the following expressions, which would best serve her purpose?

(A) (455 + 456) + (457 + 458)

(B) (647 + 373) + (227 + 456)

(C) (551 + 775) + (49 + 225)

(D) Each of these expressions would serve her purpose equally well.

Mathematical Content

The Taylor minicase focuses on the mathematical content of evaluating an expression that represents addition of three-digit numbers by applying properties of operations, not simply adding from left to right or following the order of operations. This mathematical content is represented in Grade 3 in the Common Core State Standards (Common Core State Standards Initiative, 2010), which calls for elementary school students to learn how to "fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations" (p. 24). It is notable that in this standard, the Common Core positions the properties of operations as tools that support computational fluency and not simply rules that must be memorized. Likewise, the mathematical content of the Taylor item is about being flexible in the approach to solving a problem (i.e., evaluating an expression) while at the same time retaining the structure of the quantities that are being operated on. Performing operations with multidigit numbers poses a significant challenge to elementary school students: adhering to the structure of the place-value system while achieving the goals of the operation (e.g., adding). This is why students who understand and can represent multidigit numbers and who can add single-digit numbers fluently still struggle with adding multidigit numbers. Although properties of operations are one set of strategies that can be used in solving addition problems, teachers of elementary-grade mathematics must be able to explicitly model the use of the properties with their students.

Developing this aspect of teachers' number sense is both crucial to improving mathematics instruction and a needed part of mathematics education programs (Tsao, 2005).

Student Thinking and Learning

In the assessment scenario of this minicase, Ms. Taylor's goal is to motivate student learning of the associative and commutative properties for addition by demonstrating their usefulness in solving addition problems. Ms. Taylor must select an example that will focus her students' attention on how the associative and commutative properties can be used to evaluate expressions. Third- and fourth-grade students are likely to evaluate expressions from left to right without knowing about or considering other options. Although this approach is often adequate in evaluating the kinds of basic expressions that students have seen up to fourth grade, as they are exposed to more complex mathematics, having the flexibility to add terms in an order different from the one given can improve both fluency and accuracy. Therefore, a teacher should focus on developing student awareness and use of the "any-which-way rule" for addition (Howe & Epp, 2008) or the fact that the order of addends does not impact the final sum. Developing this awareness goes beyond simply memorizing the names of operational properties, which unfortunately is what happens in some elementary mathematics classrooms. It is also important for teachers to consider how the any-which-way rule can impact understanding and application of the order of operations (e.g., parentheses, exponents, multiplication/division, addition/subtraction [PEMDAS]).

Work of Teaching

In this minicase, Ms. Taylor is determining the affordances of three potential examples for motivating her students to use the associative and commutative properties. The assumption behind this minicase is that teachers of mathematics must be able to critically assess given curricular materials from mathematical and student-learning perspectives. This may seem obvious, but teacher agency with respect to curricular materials has been, and in some places continues to be, controversial (Remillard, 2005). This minicase demonstrates how complex this work can be, as Ms. Taylor must take into account her instructional goals, the motivations of students, and the difficulty of the mathematics in choosing from among these examples.

Elaborated Answer Key

This section provides teacher educators an explanation of the answer choices for the Taylor item and a justification for the intended answer in terms of choosing the most effective example for an instructional purpose.

> A lesson in Ms. Taylor's textbook states the associative and commutative properties of addition. To motivate the students to learn the properties, she tells her students that the properties can often be used to simplify the evaluation of expressions.
>
> She wants to give her students an example that will focus their attention on how these properties can be useful in evaluating expressions. Of the following expressions, which would best serve her purpose?
>
> (A) (455 + 456) + (457 + 458)
>
> (B) (647 + 373) + (227 + 456)
>
> (C) (551 + 775) + (49 + 225)
>
> (D) Each of these expressions would serve her purpose equally well.

What Is This Assessment Item Asking?

This assessment item asks preservice teachers (PSTs) to assess three expressions for a very specific purpose—motivating elementary school students to learn the associative and commutative properties of addition. In particular, the assessment item directs PSTs to choose an example that will focus elementary school students' attention on the usefulness of the properties in evaluating expressions. This means that PSTs will need to find the expression for which application of the properties can facilitate elementary school students' fluency in addition of multidigit numbers (e.g., making the computation easier, supporting mental math).

What Information Is Important?

It is important to develop an understanding of the associative and commutative properties of addition and ways the properties can be used to help with computation. The associative property allows you to group numbers in addition in any way you would like. The commutative property allows you to reverse the relative position of two numbers when adding. Used together, these two properties allow you to add the four values in each expression in any order you like. Because this assessment item asks PSTs to choose an example that illustrates how the properties are useful, PSTs will want to pick the example that is most noticeably easier to calculate by adding the values in a different order from that which is given. One way to make calculation easier is to leverage combinations that make 10 or 100 or even 1000. This strategy is more than just a shortcut; it is an important recognition of place-value structure and is related to part/whole relationships and compensation (Fosnot & Dolk, 2001).

What Is the Rationale for Selecting an Answer?

Option A: (455 +456) + (457 +458)

Not the best example for Ms. Taylor's instructional purpose.

In this problem there are no rearrangements that lead to obvious conveniences in terms of the place value (the values in the ones places are 5, 6, 7, and 8, none of which pair to give a result of 10). Because they are consecutive numbers, knowing that the middle two values will sum to the same value as the first and last values can simplify the arithmetic a little. Therefore, this problem is made a little easier by applying the associative and commutative properties for addition, but only if you are familiar with the pattern that sums of consecutive numbers demonstrate. This option could serve Ms. Taylor's purpose, but it does not seem to be an ideal example to show elementary school students how useful the properties can be.

Option B: (647 + 373) + (227 + 456)

Not the best example for her instructional purpose.

Although this problem can associate the middle two numbers because they sum to 600, the second computation is not made easier in this case, so this will only partially simplify the computation. An example in which both parts are made simpler by rearranging would do a better job of illustrating the usefulness of the associative and commutative properties for addition.

Option C: (551 + 775) + (49 + 225)

The best example for her instructional purpose.

This problem becomes much simpler with rearrangement of the numbers. By first adding 775 + 225 to get 1000 and 551 = 49 to get 600, this problem is simplified into two quantities that can be easily summed together. Thus, this problem would very effectively highlight how the associative and commutative properties of addition can be used to make computation easier. This is the best option.

Option D

Not the best response.

Because Option C is better than Options A or B, which would not serve Ms. Taylor's purpose equally well, Option D is not the best response.

Instructional Objectives the Minicase Might Support

This section describes teacher educators' potential objectives of this minicase as a situated task to support variable instructional goals, including development of preservice teachers' (PSTs) understanding of the student-level mathematical content of expression evaluation and their practice of critically assessing given curricular materials as well as choosing an effective example for a particular instructional purpose. Although this minicase lends itself to supporting the particular objectives below, teacher educators may find additional reasons to use this case.

Understanding Student-Level Content

Fluently adding within 1000, using strategies based on properties of operations. Developing an understanding that standard conventions for order of operations, in combination with the commutative and associative properties of addition, imply an "any-which-way" rule for addition that guarantees the same sum regardless of the order of the addends or computations (Howe & Epp, 2008). The commutative and associative properties of addition allow students to choose a different, possibly "easier," way of ordering the addends than the way in which the problem is presented.

For PSTs who may be unsure why properties are important, or what students should understand about them, the problem provides a concrete example.

Developing Productive Orientations Toward K–12 Students and Student Work

Student thinking is crucial to consider at all stages of teaching, including planning. In selecting an example for classroom use, teachers should anticipate how students would view the example.

Responding to the Taylor item requires PSTs' ability to anticipate how elementary school students will make sense of the example, providing a concrete context in which PSTs can think through the connections between student thinking and example choice.

Appreciating the Larger Mathematical Idea

Seeing structure in expressions, including writing expressions in equivalent forms to solve problems; understanding when one structure may be more pertinent than another. Throughout mathematics, K–12 students can use structure to solve problems and make connections among quantities, beginning with counting and arithmetic, and later expanding to include other properties such as the associative and distributive properties.

In this item, the "right" structure to pay attention to is the commutative and associative properties of addition, although some might initially focus on the place-value structure. Discussion of why one approach makes more sense than the other may help PSTs deepen their own understanding of the properties and appreciate that instructional purpose dictates what requires attention, and in what ways.

Understanding That the Work of Teaching Requires Viewing Fundamental Mathematics Through Students' Eyes

Thinking ahead to how each of the given expressions in the Taylor item would motivate elementary school students to use the commutative and associative properties of addition requires teachers to engage in a different and more complex kind of mathematical analysis than is required to solve student-level problems.

The Taylor item could support a discussion with PSTs of what it means for a problem to motivate particular properties of addition, and why not all problems would motivate those properties equally well, making the point that examples are not equally powerful for a particular instructional purpose.

Understanding How to Select an Effective Example by Anticipating Student Thinking and Understanding

Making sense of how the commutative and associative properties for addition would be applied to each expression and how elementary school students would make sense of that work.

This item provides PSTs a context in which to practice these analytic skills, essentially serving as a practice exercise in the specific mathematics required by the situation. This might support a discussion with PSTs about what they need to know and notice to do that work, and how they might approach similar situations differently as a result of that practice.

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics.* www.amte.net/standards

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, *60*(5), 497–511. https://doi.org/10.1177/0022487109348479

Ball, D. L., Thames, M. H., & Phelps, G. C. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Barnett, C. (1991). Building a case-based curriculum to enhance the pedagogical content knowledge of mathematics teachers. *Journal of Teacher Education, 42*(4), 263–272. https://doi.org/10.1177/002248719104200404

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics*. www.corestandards.org

Fosnot, C.T., & Dolk M. (2001). *Young mathematicians at work: Constructing number sense, addition, and subtraction.* Heineman.

Howe, R., & Epp, S. S. (2008). *Taking place value seriously: Arithmetic, estimation, and algebra.* https://www.maa.org/sites/default/files/pdf/pmet/resources/PVHoweEpp-Nov2008.pdf

Howell, H., Garcia, N., & Phelps, G. (2017, March 30). *Bringing content knowledge for teaching (CKT) into the elementary mathematics teacher education curriculum* [Webinar]. American Association of Colleges for Teacher Education.

Howell, H., Lai, Y., Miller, E., & Phelps, G. (2016, January 28–30). *Assessments of secondary level mathematical knowledge for teaching as opportunities for productive struggle* [Paper presentation]. 20th Annual Conference of the Association of Mathematics Teacher Educators, Irvine, CA, United States.

Howell, H., & Mikeska, J. (2016, February 23–25). *Using content knowledge for teaching assessments as tools for learning and feedback to effect program improvement* [Paper presentation]. 68th Annual Meeting of the American Association of Colleges for Teacher Education, Las Vegas, NV, United States.

Howell, H., Phelps, G., Croft, A. J., Kirui, D., & Gitomer, D. (2013). *Cognitive interviews as a tool for investigating the validity of content knowledge for teaching assessments* (Research Report No. RR-13-19). Educational Testing Service. http://dx.doi.org/10.1002/j.2333-8504.2013.tb02326.x

Howell, H., Weren, B., & Ruiz Diaz, S. (2013, April 7–9). *Mathematical knowledge for teaching measures as opportunities for professional learning* [Paper presentation]. 2013 Annual Meeting of the National Council of Teachers of Mathematics, Denver, CO, United States.

Lai, Y., & Howell, H. (2014). Deploying tasks assessing mathematical knowledge for teaching as tasks for professional preparation. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 770–791). Research in Undergraduate Mathematics Education. http://sigmaa.maa.org/rume/RUME17.pdf

Lai, Y., & Howell, H. (2016, October 3). *Conventional courses are not enough for future high school teachers.* AMS Blogs. http://blogs.ams.org/matheducation/2016/10/03/conventional-courses-are-not-enough-for-future-high-school-teachers/

Lai, Y., Jacobson, E. D., & Thames, M. H. (2013, April 27–May 1). *The role of pedagogical context in measures of specialized and pedagogical content knowledge* [Paper presentation]. 2013 Annual Meeting of the American Educational Research Association, San Francisco, CA, United States.

Phelps, G., & Howell, H. (2016). Assessing mathematical knowledge for teaching: The role of teaching context. *The Mathematics Enthusiast*, *13*(1–2), 52–70.

Phelps, G., Howell, H., Weren, B., & Ruiz Diaz, S. (2013, January 24–26). *Assessments of content knowledge for teaching (CKT) as opportunities for teacher learning* [Paper presentation]. 17th Annual Conference of the Association of Mathematics Teacher Educators, Orlando, FL, United States.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(11), 211–246. https://doi.org/10.3102/00346543075002211

Shulman, J. H. (Ed.). (1992). *Case methods in teacher education.* Teachers College Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14. https://doi.org/10.3102/0013189X015002004

Stake, R. E. (1987). *An evolutionary view in programming and staff development.* In M. F. Wideen & I. Andrews (Eds.), Staff development for school improvement: A focus on the teacher (pp. 55–69). Falmer Press.

Sykes, G., & Bird, T. (1992). Teacher education and the case idea. *Review of Research in Education, 18*(1), 457–521. https://doi.org/10.3102/0091732X018001457

Tsao, Y. L. (2005). The number sense of pre-service elementary school teachers. *College Student Journal*, *39*(4), 647–679.

Appendix A. Sample Lesson Outline

This appendix provides teacher educators a sample lesson outline, including lesson goals, links to prior learning of and about operational properties, and suggestions for lesson implementation to use with preservice teachers (PSTs). This sample lesson may provide an illustration of how a whole lesson can be planned around the Taylor minicase, and is designed to be user ready, although it is only one example of how a lesson might be configured.

Using Properties

Goals for This Lesson

For preservice elementary-school teachers to. . .

- Understand that the work of teaching requires the ability to view fundamental mathematics through students' eyes, particularly when selecting an example to make an instructional point.
- Recognize that effectively choosing examples involves anticipating student thinking (in this case, how students think about the associative and commutative properties of addition).
- Recognize the connection between example choice and instructional goal and learn how to select examples based on variable instructional goals.
- Understand that the operational properties are to be *used* (i.e., not merely memorized), and be able to articulate reasonable student learning goals around the associative and commutative properties of addition.
- Discuss the "any-which-way" rule for addition (Howe & Epp, 2008), what underlying mathematics it captures, and the degree to which formalizing or "informalizing" language is appropriate for elementary-school students.
- Think through how one might encourage elementary-school students to develop an "any-which-way" rule for addition, define it, and decide when it works and when it doesn't.

Embedded Student Content

In this lesson, PSTs are to assess the following expressions in terms of their usefulness in motivating elementary-school students to use the commutative and associative properties of addition:

(455 + 456) + (457 + 458)
(647 + 373) + (227 + 456)
(551 + 775) + (49 + 225)

The key to this task is to realize that although computation of the first two options can be somewhat simplified using the associative and commutative properties of addition, in the third option, computation is made much easier by adding 551 + 49 and 775 + 225 because this results in 600 and 1000.

Lesson Overview

Through engaging with the problems above, PSTs will work to recall their knowledge of operational properties and their use.

Warm-Up

Ask PSTs to take 5 minutes to evaluate the three expressions above. The purpose of this exercise is to prime PSTs' own approaches to evaluating an expression and the reasoning behind their approaches to evaluating an expression. It can also give you an idea about PSTs' prior learning and understanding of properties of operations and their use. Once PSTs have finished evaluating, you can ask a few of them to share how they have done so, or poll the class to see, for example, how many

- added from left to right;
- computed the additions inside the parenthesis first by following the order of operations; and/or
- reorganized the addends to get easier numbers to compute by applying the commutative and associative properties of addition to simplify calculations, etc.

Regardless of the specific PST responses, this brief discussion can serve to establish that

- PSTs may have used different approaches for evaluating the expressions. Although all approaches reach the same sums for each of the expressions, one approach can be more efficient for simplifying computation than others for the given problem context.
- When evaluating an expression using the commutative and associative properties of addition, it doesn't matter which addition is done first (i.e., implying the "any-which-way rule" for addition [Howe & Epp, 2008]).
- The most critical point about evaluating an expression using the commutative and associative properties of addition is that the addends in the expression can be rearranged flexibly to make the computation easier. Flexibility (i.e., being free from the order of operations) will help to increase fluency of addition of multidigit numbers.

Acknowledging the use of the commutative and associative properties of addition as an efficient approach to simplify computation while evaluating an expression and promoting the use of these properties provides a segue to the rest of this lesson.

Solving the Taylor Item

Provide the Taylor item to PSTs and ask them to solve it in groups or individually. As they complete this item, you may want to note

- Are the PSTs still concerned with other rules such as the order of operations in evaluating an expression, or are they looking for ways the commutative and associative properties of addition can be used to simplify computation?
- Are PSTs able to identify Option C as the expression that best shows how the commutative and associative properties of addition can be used to simplify computation?

Discussion of the Taylor Item

Bring the whole class back together to discuss how they solved the Taylor item. If you have access to any audience-surveying tools, you can use them to poll the class anonymously on their selection of Option A, B, C, or D. Otherwise, you can take a quick poll of which PSTs chose which options and ask volunteers to explain their thinking. Note that it may be best to choose A, B, or C before or instead of D because Option D accepts all of the previous responses. When discussing this item with PSTs, you may want to listen and prepare for, or even raise, the following discussion points:

"Any of the problems would work because the commutative and associative properties of addition are always true." It is the case that the properties always hold; that's part of what makes them important to learn and to teach. So the commutative and associative properties of addition can absolutely be applied to any of the given problems. However, Ms. Taylor's challenge is to convince elementary-school students that the properties are useful. If we assume that what will motivate elementary-school students is simplifying calculations, then Option C is the best example for Ms. Taylor to choose.

"None of these problems illustrates the commutative and associative properties of addition because the expressions aren't written in the form "this + this = that." Some PSTs may believe that commutative and associative properties of addition apply only to equations (and not to expressions). This is likely because PSTs as students may have learned the properties in the context of equations only (e.g., in addition, the order of two values can be transposed without changing the sum). It is important for PSTs to know that the commutative and associative properties of addition can be used in various contexts, and it is the use that is arguably most important for elementary-school students to learn.

Some PSTs may propose additional ways of using the commutative and associative properties of addition along with de- or recomposing the values given. They may see that the way these properties are used is to allow you to group by place value. It's a really important connection to make that these properties do support decomposition of numbers and representation in a place-value system, and that is an incredibly important structure in mathematics. However, elementary-school students are unlikely to (a) understand fully what this means or (b) see that this is a huge affordance. For students in this grade range, an example that clearly makes calculation easier and faster is more likely to be convincing.

In summarizing the discussion, it is important to emphasize that the commutative and associative properties of addition, taken together, allow you to rearrange the 3-digit numbers any-which-way for addition, which is to say, the commutative and associative properties of addition let you arrange the addends in ways that simplify calculation. Elementary-school students are likely to find simplification of calculation highly motivating and to see this as something useful the commutative and associative properties of addition do for them. They are also likely to understand this in slightly imprecise ways, such as "you can rearrange the order," and not clearly distinguish between the two properties. As adults, some PSTs may recognize that they use these properties when having to add a short sequence of numbers in everyday life.

PSTs should see that it's really important to consider in advance what their students are likely to see in, appreciate about, and take away from an example. This also means that if their instructional goals are different from those of Ms. Taylor, they might choose a different expression from among the three offered. Therefore, there is no one "best" example problem because it depends what PSTs are trying to communicate and to whom. For example, Ms. Taylor (and the PSTs) could also consider

- How precisely should students understand the commutative and associative properties of addition individually and together?
- What are the pros and cons of using less formal language to motivate use of the commutative and associative properties of addition?

How does this rearranging idea relate to PEMDAS? Although PEMDAS is a way to remember how different operations interact, it doesn't address how each of these operations can be used flexibly. Flexibility will help elementary-school students develop number sense, do mental math, and increase fluency of addition of multidigit numbers. Although the elementary-school students presented in the Taylor item may not have yet been introduced to PEMDAS, these ideas are important for PSTs to understand.

Discussion of the Taylor Item

There are several ways that the ideas mentioned above can be reinforced and applied with a homework assignment. For example, PSTs can

- Read Howe and Epp's 2008 article that conceives the commutative and associative properties of addition as forming an any-which-way rule for addition; and
- Choose a different instructional purpose (i.e., not illustrating the usefulness of the properties) for which PSTs might choose a different expression from among the three options. In other words, rewrite the item text so that there is a different right answer. PSTs should write down (a) the text; (b) what they're choosing as an example to

illustrate; (c) what it is they anticipate elementary-school students will understand from the example; and (d) which answer choice is the correct one given the new problem context.

Appendix B. Additional Discussion Prompts

This appendix provides teacher educators more ideas to prompt and orchestrate discussion with preservice teachers (PSTs) with regard to the Taylor item in addition to the sample lesson plan for the Taylor minicase.

The following is a list of potential discussion prompts, extensions, or additional assignments teacher educators might use with PSTs around the Taylor item. Have the PSTs think about

- Why are the commutative and associative properties of addition part of the K–12 mathematics curriculum?
- What is important for elementary-school students to know about the commutative and associative properties of addition?
- For each of the three expressions in the item, describe what an elementary-school student might do to evaluate the expression. If possible, have 2–3 elementary-school students evaluate the expressions in this item to compare across expressions and/or across students.
- What follow-up question(s) might you pose to an elementary-school student if you wanted to help the student see how the commutative and associative properties of addition can be useful in evaluating an expression?
- Create a similar problem in which elementary-school students' approaches to evaluating an expression would be noticeably easier when using the commutative and associative properties of addition.
- Having PSTs reflect back on their own work on the three expressions from the Taylor item, what different instructional purposes can be set for each of the expressions (other than motivating students learning to use the commutative and associative properties of addition)?

Appendix C. Resources and References

This appendix provides additional resources that are relevant to the mathematics and/or teaching practices mentioned in this minicase. In particular, the article from the *Mathematics Teacher,* below, can be potentially assigned as reading for preservice teachers (PSTs).

Howe, R. & Epp, S. S. *Taking place value seriously: Arithmetic, estimation, and algebra.* https://www.maa.org/sites/default/files/pdf/pmet/resources/PVHoweEpp-Nov2008.pdf

Larsen, S. (2010). Struggling to disentangle the associative and commutative properties. *For the Learning of Mathematics*, *30*(1), 37–42.

Olson, J. R. (2015). Five keys for teaching mental math. *Mathematics Teacher*, *108*(7), 543–547. https://doi.org/10.5951/mathteacher.108.7.0543

Tucker, A. (n.d.). *Place value and integer arithmetic: A study from first principles*. http://www.ams.sunysb.edu/~tucker/placevalue

Appendix D. Frequently Asked Questions

Where did the assessment items come from?

These items were originally written for use in the Measures of Effective Teaching (MET) Project: https://k12education.gatesfoundation.org/blog/measures-of-effective-teaching-met-project/ as part of an assessment of content knowledge for teaching. The team that wrote the items included researchers at the University of Michigan and Educational Testing Service (ETS). At the conclusion of the study, the items were released for use, with some restrictions. Copies of the assessment forms can be requested from the ETS lead for the MET study, Geoffrey Phelps, gphelps@ets.org.

There's something I would like to change about the item. / I don't agree with the way the math is presented in the item. Would you consider changing it?

We decided in our work on the minicases to use the assessment items exactly as they were provided by the projects they came from. One goal of the further development work is to explore how existing intellectual capital in the form of assessments can be repurposed into material for teacher learning. The minicases have developed organically across a set of projects over a number of years, and there have been many contributors to them. The latest versions were reviewed by four experts in the field of mathematics and mathematics education, and their advice has been incorporated into revisions.

Part of what we want to illustrate is that the assessment item itself need not be above critique for it to be a useful starting point for PST learning. In fact, we think some critique might signal rich points for discussion as part of teacher development. That said, the point of the minicase is to be provocative, not prescriptive, and we encourage anyone who wishes to tweak, alter, subvert, delete, or completely rewrite the assessment item in service of their own instructional goals to do so. (And if it's an item from the Google drive, we hope you'll post your work back in the drive for others to use!)

I would like to use these items as a hiring screen for new teachers; where could I find more of them?

This is not an approved use of these items. Accessing these items requires that you agree to terms of use that exclude high-stakes decision making.

Where could I find more minicases like these?

We have only a few exemplars ready for use at the current time, but we are more than happy to share them on request. To be added to our distribution list, contact Heather Howell, hhowell@ets.org. The minicases are a work in progress. If you have suggestions, please let us know!

Notes

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