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Ms. Williams's Analysis: An Instructional Minicase on Interpreting Student Work About the Simplification of Exponential Expressions

Heather Howell
Yvonne Lai
Leslie Nabors Oláh
Maria DeLucia
Eun Mi Kim

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Heather Howell
ETS, Princeton, New Jersey, United States

Yvonne Lai
Department of Mathematics, University of Nebraska – Lincoln

Leslie Nabors Oláh
ETS, Princeton, New Jersey, United States

Maria DeLucia
Department of Mathematics, Middlesex Community College

Eun Mi Kim
ETS, Princeton, New Jersey, United States

October 2021

Corresponding author: H. Howell, E-mail: hhowell@ets.org

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Abstract

There is a broad consensus that beginning teachers of mathematics need a strong foundation in mathematical knowledge for teaching (MKT), defined as the mathematical knowledge required to recognize, understand, and respond to the mathematical work of teaching in which one must engage. One recurrent challenge in teacher education is how to provide support for preservice teachers (PSTs) to acquire such competencies. Recent trends toward practice-based teacher education support the idea of engaging novice teachers in activities that are purposefully constrained to a core teaching practice. Ms. Williams's Analysis is an abbreviated instructional case (i.e., a minicase) based on an assessment scenario in which PSTs are asked to attend to two students' application of the laws of exponents in the course of simplifying an exponential expression. PSTs are asked to judge the mathematical validity of the students' explanations as a way of further developing their own MKT.

Keywords: mathematics education, exponential expressions, student explanations, mathematical knowledge for teaching (MKT), teacher preparation, preservice teacher (PST)

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We plan to increase the number of minicases in the coming years and to make further improvements based on feedback from those using the materials. If you would like to make suggestions, please contact Heather Howell at hhowell@ets.org.

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Maria DeLucia is now retired from Middlesex Community College.

On the following pages we present the fruits of a line of work that has spanned multiple projects over multiple years and reflects the contributions of a number of individuals at different points in time. The rationale for the minicases' development is, in essence, quite simple. Much of recent scholarship on teachers' mathematical knowledge for teaching (MKT) has focused on the assessment of MKT via practice-based questions. Practice-based questions generally include a short introductory scenario whose features are critical in solving the task. These scenarios are not simply window-dressing for the task but rather, along with the specified mathematical content, they codefine what is measured (Phelps & Howell, 2016). As such, these tasks can be understood to constitute abbreviated representations of teaching practice (Lai et al., 2013).

Because there has been intense interest in the field in assessing MKT, sample assessment tasks currently make up much of the field's description of specific MKT. Since such assessments became available, we have been approached by several teacher educators interested in integrating MKT assessment items into the curricular content of their mathematics and mathematics methods courses, not by using them as assessments, but by using them as exemplar instructional cases rather than assessment (see Lai & Howell, 2014, for example tasks). However, a number of obstacles to this kind of use have been noted, leading teacher educators to request publicly available full sets of materials that are aligned to instructional goals. Our goal in developing the minicases was to take on some of these challenges by developing a set of support materials designed to aid teacher educators in making use of the items as a curricular resource and, at the same time, illustrate one type of support that could be developed more broadly out of such items.

The development team consists of researchers in mathematics and mathematics education, as well as current teacher educators. This work began initially as part of a 2011 project at ETS intended to investigate the design features of MKT items in hopes of identifying relationships between structural features of the items and how well they performed in measuring MKT. This project used released items from the Measures of Effective Teaching (MET) Project (Bill and Melinda Gates Foundation, 2013) and, for each item, created an analytic memo, the purpose of which was (a) to document the reasoning a test taker might use in responding, clearly identifying in each case not just why the intended answer was best but also the logical basis on which each of the competing answer choices could be discarded, and (b) to map that reasoning to types of specialized, common, and pedagogical knowledge as described in Ball et al.'s (2008) theory of MKT. Over the subsequent year, the team worked to refine these documents and tailor them to the possibility of serving multiple audiences, including item writers, researchers, teacher educators, and test takers themselves. We used this documentation in a validity study (Howell, Phelps, et al., 2013) and disseminated it at a number of conferences (Howell et al., 2016; Howell et al., 2017; Howell & Mikeska, 2016; Howell, Weren, & Ruiz Diaz, 2013; Lai & Howell, 2014; Phelps et al., 2013) where we received not only critical feedback but also enthusiastic reception from teacher educators eager to see and use more of them. In 2013, a separate project funded by the National Science Foundation (NSF; https://www.nsf.gov/awardsearch/showAward?AWD_ID=1445630) created a set of secondary level MKT items with accompanying documentation, collected similar validity evidence (Lai & Howell, 2016), and furthered our dissemination goals by creating a Google group in which the items and documentation are housed and available to interested parties (see Appendix D).

With a critical mass of systematic assessment documentation at hand, we decided to further develop this material into a set of "MKT minicases," documents designed to be used directly by teacher educators in supporting preservice teachers' (PSTs) development of MKT. We chose the name *minicase* to distinguish these materials from *instructional cases* (L. S. Shulman, 1986; Stake, 1987) because they differ from each other in structure and in degree of specificity (J. H. Shulman, 1992). The minicases are shorter than many cases used in professional preparation and are not structured to reveal additional information beyond the initial scenario. The minicases also target very specific knowledge about teaching and learning and are less open to interpretation than most instructional cases.

In 2016 and 2017, ETS funded the development of four minicases (two at elementary level and two at secondary level) based on teacher educator input. In 2018, we solicited reviews of the materials from four researchers in the fields of mathematics and mathematics education and six practicing teacher educators. The feedback from these reviews was then used to revise the set of four minicases to improve mathematical accuracy and comprehensiveness, as well as usability.

Background

There is a broad consensus that beginning teachers of mathematics need a strong foundation in mathematical knowledge for teaching (MKT), defined as the mathematical knowledge required to recognize, understand, and respond to the mathematical work of teaching in which one must engage (Ball et al., 2008). Standards call out, for example, competencies for beginning teachers such as possessing “robust knowledge of mathematical and statistical knowledge and concepts, . . . expanding and deepening [PSTs’] knowledge of students as learners of mathematics,” and engaging in “effective and equitable mathematics teaching practice” (Association of Mathematics Teacher Educators, 2017, p. 6). One recurrent challenge in teacher education is how to provide support for PSTs to acquire such competencies. Recent trends toward practice-based teacher education support the idea of engaging novice teachers in activities that are purposefully constrained to a core teaching practice (Ball & Forzani, 2009). The MKT minicases we have developed represent one such example.

Research on using cases for subject-specific teacher learning goes as far back as the 1990s (Sykes & Bird, 1992). In mathematics and teacher education, cases can also provide a common language, explicit expectations of high-quality mathematics teaching, information about K–12 student development and common misunderstandings, and a means to interact with challenging content (Barnett, 1991).

Each minicase includes a situated task of teaching practice originally developed as part of teacher assessment efforts. Our guiding hypothesis is that these assessment scenarios, along with the accompanying documents that make up the minicases, form a set of resources for teacher educators. These resources are designed to support instructional goals, including developing PSTs’ understanding of K–12 student and higher level mathematics, developing PSTs’ orientations toward K–12 students and student work, helping PSTs understand what makes up the professional work of teaching mathematics, and providing them opportunities to engage in the cognitive work associated with addressing the given task.

Because each situated task was originally designed for assessment purposes and crafted to have a single best answer, the resulting minicases require users to take a stand with respect to the presented problem. These cases, unlike instructional cases that are more open-ended, invite response and disagreement in a way that can support rich but focused discussion. Our intention is to support teacher educators who are teaching math methods courses or math content courses for PSTs by providing a set of materials that can be used flexibly and adapted as appropriate.

Instructional Task: The Williams Item

Ms. Williams is reviewing a set of homework problems in which students were asked to evaluate exponential expressions, including the expression $\left((-9)^{\frac{1}{2}}\right)^2$. Ms. Williams asks two students to share their work. For each of the following students' work, indicate whether it demonstrates a valid application of the laws of exponents to solve the problem.

Student work	Valid application	Not valid application
Craig said: I used the exponent rule to change the order of the squared and the one half because that way you don't have to take the square root of a negative number: $(-9)^2 = 81$. Then the square root of 81 is 9.		
Katlynn said: I did it an easier way. 2 and $\frac{1}{2}$ cancel, so it's just -9.		

Mathematical Content

The Williams minicase focuses on the simplification of exponential expressions as well as the practices of interpreting student work and evaluating mathematical reasoning. Exponents are a core inclusion in most curricula, generally consisting of an introduction to the notation and calculation of exponents around Grade 6 or 7 and a generalization to working with negative and rational exponents between Grades 7 and 9 and exponential functions in the real number system as the highest level generally reached in secondary curricula, usually in Algebra 2 or precalculus (Common Core State Standards Initiative, 2010). The secondary curriculum does not usually address complex numbers, either as bases or as exponents. High school students encounter the exponential function, again usually in Algebra 2 or precalculus, which is often directly related to their previous knowledge of exponents in terms of calculations (e.g., filling in a table of values). A typical initial way of defining exponentiation at the secondary level is as repeated multiplication, and this definition is seldom revisited.

The Common Core State Standards (Common Core State Standards Initiative, 2010) reflect a strong focus on using established properties to simplify exponential expressions, specifying that K–12 students should "use whole-number exponents" (p. 35), "evaluate numerical expressions involving whole-number exponents" (p. 43), "perform arithmetic operations, including those involving whole-number exponents" (p. 44), and "know and apply the properties of integer exponents" (p. 54). The standards also suggest that a layered extension of exponent definitions is characteristic of a type of expansion of number systems seen throughout mathematics (Common Core State Standards Initiative, 2010, p. 58) and models a system consistency argument in the definition of rational exponents:

> Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5. (p. 60)

PSTs are generally conversant with the arithmetic involved in simplifying exponential expressions. However, they may be less familiar with the nuances involved with extending the definition of exponents and proofs of the common properties of exponents beyond positive bases and integer exponents and may, in fact, be unaware that the properties do not extend seamlessly in all cases. PSTs need opportunities to learn this mathematics in order to avoid the types of pitfalls showcased in the Williams item and to understand that this content constitutes an opportunity for high school students to learn about domain restriction and extension as mathematical activities.

We also note that while the complex numbers present in this particular problem might also be referred to as imaginary numbers since they have no nonzero part, we use the more general term, *complex numbers*, throughout for mathematical clarity.

Student Thinking and Learning

In the Williams item, high school students have been asked to simplify the expression $\left((-9)^{\frac{1}{2}}\right)^2$. There are a number of properties of arithmetic at play that high school students might be likely to consider. A standard order of operations might suggest that simplifying "inside-out" is the appropriate approach, by first taking the square root of -9, then squaring the result. However, high school students may or may not know how to take the square root of a negative number. If working only in the real number system, "undefined" is a reasonable answer, although high school students often simply use the word to indicate that a calculation is impossible. Often this statement is made without conceptual understanding that there is, in fact, a definitional issue at play. High school students are also likely familiar with properties of exponents that are often used to simplify calculation in such cases, including the power of a power property for exponents, $(x^a)^b = x^{ab}$, and may apply this property to find a “short cut” in a number of different ways, including cancelling the exponents by noting their inverse relationship or commuting the exponents. In other words, a likely high school student approach is to utilize exponential properties to avoid the definitional issue embedded in the given expression and produce a numeric answer, which the high school student may see as more satisfying than reporting that the expression is undefined. A number of potential negative outcomes could result from this approach. A novice teacher might encourage incorrect application of properties because it produces a “correct” answer, essentially falling into the same trap as a high school student might, and may unintentionally reinforce the notion that properties established over one domain can be used without consideration in broader domains. PSTs also risk losing a pedagogical opportunity; few mathematical topics in the K–12 curriculum afford clear examples of problematic extension from one domain to another, and the case of exponents constitutes one that can be used to illustrate this possibility and motivate careful attention to domain specification.

Work of Teaching

The Williams item asks PSTs to interpret student work and explanations and to evaluate the mathematical reasoning in terms of how the properties of exponents have been applied, not just in terms of whether a “correct” answer was reached. While these skills will continue to develop throughout a teacher’s career, introducing PSTs to these expectations now will improve their beginning instruction. In particular, recognizing that high school students have implicitly invoked the power property for exponents is critical for PSTs to notice, as is attention to the question at hand (application of properties), more so than recognizing whether they have reached a correct numerical result.

Elaborated Answer Key

This section provides teacher educators an explanation of the answer choices of the Williams item and a justification for the intended answer in terms of a mathematical validity and generalizability.

What Is This Assessment Item Asking?

This assessment item asks PSTs to determine, for each high school student, whether the given work constitutes a valid application of the laws of exponents to solve the problem. It is important to note that the focus is on the validity of the application of laws of exponents, not on the arrival at a correct answer. To answer this assessment item PSTs need to consider the approach each student has taken, determine what laws of exponents are likely to have been invoked, and decide whether the uses of the laws are valid for solving the problem.

What Information Is Important?

The most important issue to consider is the generalizability of the properties of exponents. In K–12 mathematics, exponents are often initially defined in terms of repeated multiplication where the bases are assumed to be positive real numbers and the powers are assumed to be positive integers. Properties of exponents are often proven using techniques that depend on these assumptions, most importantly the assumption that the powers are positive integers. For example, the power property $(x^a)^b = x^{ab}$ is often proven by relating exponentiation to repeated multiplication:

> $(x^a)^b$ means multiplying together *b* copies of x^a, where x^a means multiplying together *a* copies of *x* so this yields a total of *ab* copies of *x*, which is x^{ab}.

Later in the curriculum, both the domain of the bases and the domain of the powers are extended. If the domain of the powers is still positive integers, then we can extend the domain of the bases to include all real and even complex numbers. However, extending the domain of the powers to include zero, negative integers, rational numbers, or all real numbers requires refining the definition for exponents. For these domains, exponentiation is no longer defined strictly in terms of repeated multiplication, so the previously established properties of exponents cannot be assumed to hold if their proofs depended on repeated multiplication. In many cases, the common properties of exponents can be extended in ways that are consistent and do not lead to problems. But in some cases, they cannot. Specifically, the power property $(x^a)^b = x^{ab}$ can be problematic when $x < 0$. For example, in the given problem the order of computation of the exponents will matter because one order involves imaginary numbers whereas the other does not:

$$\left((-9)^{\frac{1}{2}}\right)^2 = (3i)^2 = -9$$

but

$$((-9)^2)^{\frac{1}{2}} = (81)^{\frac{1}{2}} = 9.$$

In other words, $((-9)^2)^{\frac{1}{2}} \neq \left((-9)^{\frac{1}{2}}\right)^2$ either because $((-9)^2)^{\frac{1}{2}} \neq (-9)^{\frac{1}{2}\cdot 2}$ or because $\frac{1}{2} \cdot 2 \neq 2 \cdot \frac{1}{2}$. Since real numbers do commute, we have to conclude that the power property does not hold.

A second complication inherent in the given problem is the question of what domain the students are working over. Specifically, if the students are working over the complex numbers, it can be argued that the given expression $\left((-9)^{\frac{1}{2}}\right)^2$ has a single right answer, which is -9. If the students are working over the real numbers, it can be argued that the given expression must be considered to be *undefined* (i.e., does not have a meaningful mathematical interpretation) because if a given set of symbols in an expression is undefined, performing additional operations on that object is not possible.

What Is the Rationale for Selecting an Answer?

Craig uses the properties of exponents to reverse the order of calculation in the expression. Although not specified in detail, doing so would likely make use of properties of exponents as shown:

$\left((-9)^{\frac{1}{2}}\right)^2 = (-9)^{\frac{1}{2}\cdot 2}$ uses the power property: $(x^a)^b = x^{ab}$

$= (-9)^{2\cdot\frac{1}{2}}$ applies commutativity property of real numbers: $(x^a)^b = x^{ab}$

$= ((-9)^2)^{\frac{1}{2}}$ uses the power property: $x^{ab} = (x^a)^b$.

This is a clever move on Craig's part, as he is using a rule he believes to be true to make the calculation easier to execute. As discussed, however, the power property does not hold in this case, and his application of it is not valid. His application of the properties also leads to an incorrect answer. We do not know whether these students are working over the complex numbers, but if we assume that they are, the correct answer to the problem is -9. If we assume that they are not, it would be correct to state that the expression is undefined.

Although Katlynn is likely not thinking in terms of properties of exponents, she implicitly uses the same property of exponents that Craig did:

$\left((-9)^{\frac{1}{2}}\right)^2 = (-9)^{\frac{1}{2}\cdot 2}$ uses the power property: $(x^a)^b = x^{ab}$

$= (-9)^1$ evaluates: $\frac{1}{2}\cdot 2$

$= -9$ uses the property: $x^1 = x$

Her application of the power property is problematic, but unlike Craig's, coincidentally leads to a correct answer. But this does not mean that it would work in all cases where x is negative and the exponents are rational. For example, if the given expression had been $((-9)^2)^{\frac{1}{2}}$, Katlynn's method would have still produced the answer -9, which would then be incorrect.

Both students apply the power property to a situation that is not appropriate, and neither approach is mathematically valid. Therefore, the intended answer to the item is as follows:

Student work	Valid application	Not valid application
Craig said: I used the exponent rule to change the order of the squared and the one half because that way you don't have to take the square root of a negative number. $(-9)^2 = 81$ Then the square root of 81 is 9.		✓
Katlynn said: I did it an easier way. 2 and $\frac{1}{2}$ cancel, so it's just -9.		✓

Instructional Objectives the Minicase Might Support

This section describes teacher educators' potential objectives of this minicase as a situated task to support variable instructional goals, including development of PSTs' understanding of the mathematical content of exponents and their properties as well as their practice of interpreting and evaluating student work in terms of mathematical validity and generalizability. This minicase lends itself to supporting the particular objectives below; however, teacher educators may find additional reasons to use this case.

Understanding Student-Level Content

Student content includes evaluating exponential expressions that are potentially undefined, accounting for domain restrictions in the use of algebraic manipulations of expressions, and understanding the limitations they introduce. In also includes attending to and articulating underlying assumptions about properties of exponents and their domains. For PSTs who may not have explicitly encountered the dependence of a mathematical expression on the assumed domain of operation or who are not familiar with this nuance in the context of exponents, examining the student work provides a concrete example that allows discussion around the mathematics.

Developing Productive Orientations Toward K–12 Students and Student Work: Emphasizing the Practice of Interpreting Student Explanations

An important orientation in solving the Williams item is recognizing the importance of student explanations, as answers alone can mask potential understandings and misunderstandings. To respond to the Williams item coherently requires PSTs' analysis of the given student explanations, not just the answers the students provided. This experience could give PSTs a concrete context in which to discuss more general dispositions or instructional values, such as giving K–12 students opportunities to explain, listening carefully to student explanations, and evaluating their responses in terms of underlying mathematics.

Appreciating the Larger Mathematical Idea

A larger idea at play in the Williams item is that domain specification, often left implicit in grade-school level mathematics, is a key part of constructing mathematical definitions.

Extension of definitions and properties is a common mathematical activity that is desirable as a way to advance new ideas and potentially problematic. The item context provides PSTs a place to discuss the importance of domain and the mathematical practice of domain extension. It illustrates how a fundamental mathematical principle can play out in school-level mathematics and how such a case might constitute an opportunity to learn these principles.

Understanding That the Work of Teaching Requires Close Analysis of Student-Generated Work and Attention Both to the Validity of the Underlying Mathematics and to the Adequacy of the Explanation

Analyzing student-generated strategies requires teachers to engage in a different and more complex kind of mathematical analysis than is required to solve the student-level problems. Responding to the item correctly requires that PSTs understand the students' perception of how they solved the problem by identifying the properties they used and determining whether those properties hold. Discussion of this minicase provides a concrete illustration of how a relatively simple mathematics problem can head "into the deep end" mathematically and may serve as a caution against taking simple algebraic manipulation at face value.

Understanding How to Analyze Student Work Samples in Terms of the Validity and Adequacy of the Explanations: Making Sense of the Student Work in This Item

The minicase provides opportunities for the PST to interpret student work in terms of what the students believe they have done, to explore the properties invoked in those algebraic moves, and to unpack the underlying mathematics that justifies those algebraic moves. The item provides PSTs a context in which to practice these analytic skills, serving as a practice exercise in the mathematics specific to the minicase.

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Appendix A: Sample Lesson Outline

This appendix provides teacher educators with a sample lesson outline, including lesson goals, links to prior learning of and about exponential properties, and suggestions for lesson implementation to use with PSTs. This lesson is expected to take approximately 3 hours. If teaching in two sessions, teacher educators might pause the lesson after introducing the Williams item and before discussion about the high school students' work. This sample lesson may provide an illustration of how a whole lesson can be planned around the Williams minicase and is designed to be user-ready, although it is only one example of how a lesson might be configured.

Domain of Exponential Properties

Goals for This Lesson

The goals of this lesson are for preservice middle and high school teachers to

- articulate and attend to underlying assumptions about properties of exponents and their domains;
- understand that the value of a mathematical expression, as undefined or not, depends on the domain of mathematical operations used to interpret the expression and that the domains of operations are extended over time to learners; and
- analyze how middle and high school students are using exponential properties and what assumptions they are implicitly drawing upon.

Embedded Student Content

In this lesson, PSTs are asked to analyze two high school students' work on the given task:

Evaluate the expression $\left((-9)^{\frac{1}{2}}\right)^2$.

The task asks the high school students to find the value of this exponential expression. Prior to learning complex numbers, one approach to solving the task is to notice that $(-9)^{\frac{1}{2}}$ is undefined because "negative numbers do not have square roots," and therefore the entire expression must be undefined. However, if complex numbers are known, there is actually still a subtlety about the inner expression, for *i* is only defined as a value such that $i^2 = -1$. In this sense, we cannot distinguish between *i* and −*i*, so the value of $(-9)^{\frac{1}{2}}$ is $\pm 3i$ if we do not declare any conventions about which branch of the square roots the $\frac{1}{2}$ exponent denotes. (In the case of positive numbers, we say by convention that $\sqrt{x}$ or $x^{\frac{1}{2}}$ always denotes the positive root; we can adopt a similar convention for negative numbers by saying that it always denotes the positive multiple of *i*.) Whatever the convention we use, $(-3i)^2 = (+3i)^2 = -9$, so we may conclude that $\left((-9)^{\frac{1}{2}}\right)^2 = -9$.

Opener: Evaluating Exponential Expressions

This lesson begins with a task to elicit PSTs' thinking about exponentiation and arithmetic properties and how they work:

> Find as many different ways as possible to find the value of $\left((7)^{\frac{1}{2}}\right)^2$. What properties (you may think of them as facts or rules) about exponentiation and arithmetic did you use?

Discussion for Opener on Evaluating Exponential Expressions

The purpose of this discussion is twofold:

1. for PSTs to see and name commonly used properties of exponentiation; and
2. for PSTs to see that these properties have surprising results when applied outside standard domains.

The examples in the opener provide a way to see why the standard domain for bases is positive real numbers and how the properties seem to break down when the base is negative and the exponents are not integers.

As the PSTs are working on this problem, note the strategies they have used. To begin the discussion, write some strategies on the board, including all from the table below even if they have not come up from the PSTs' own work.

Possible strategies (numbered for convenience) include the following:

1. $\left((7)^{\frac{1}{2}}\right)^2 = 7^{\left(\frac{1}{2}\right)\cdot 2} = 7^1 = 7.$

2. $\left((7)^{\frac{1}{2}}\right)^2 = 7^{\left(\frac{1}{2}\right)\cdot 2} = 7^{2\cdot\left(\frac{1}{2}\right)} = ((7)^2)^{\frac{1}{2}} = \sqrt{49} = 7.$

3. $(7)^{\frac{1}{2}}$ is the square root of 7. When you square a square root, you get back the original number. So when you square $(7)^{\frac{1}{2}}$, you get 7.

For each strategy on the board, ask the PSTs: "This is a strategy that came up. What might be the reasoning behind this strategy? What properties or definitions does each step use?"

This might lead to a table such as this one:

Strategy	Reasoning Behind This Strategy
$\left((7)^{\frac{1}{2}}\right)^2 = 7^{\left(\frac{1}{2}\right)\cdot 2} = 7^1 = 7$	$(x^a)^b = x^{ab}$, power of a power property
$\left((7)^{\frac{1}{2}}\right)^2 = 7^{\left(\frac{1}{2}\right)\cdot 2} = 7^{2\cdot\left(\frac{1}{2}\right)} = ((7)^2)^{\frac{1}{2}}$ $= \sqrt{49} = 7$	$(x^a)^b = x^{ab} = x^{ba} = (x^b)^a$ 1. power of a power property 2. commutative property of multiplication 3. power of a power property This strategy also uses the convention that $\sqrt{x}$ denotes only the positive square root.
$(7)^{\frac{1}{2}}$ is the square root of 7. When you square a square root, you get back the original number. So when you square $(7)^{\frac{1}{2}}$, you get 7.	Definition of square root and square Convention that $x^{\frac{1}{2}}$ (and $\sqrt{x}$) denote the positive square root (not the negative square root).

Key points to surface from the discussion are the following:

- The power of a power property works in two different directions. You can either go from x^{ab} to $(x^a)^b$ or go from $(x^a)^b$ to x^{ab}.
- It was possible to use the commutative property of multiplication only after using power of a power property to create an expression with only one exponent.

- Typically, $(x)^{\frac{1}{2}}$ and $\sqrt{x}$ denote only the positive square root of a number. They do not denote the negative square root. However, the square root of a number is defined as any number that squares to that number.

Situating the Concepts in Teaching: Ms. Williams's Class

This case sets up the central mathematical inquiry for this lesson: When is an exponential expression well-defined? By well defined, we mean that a numerical expression can be evaluated to be a unique numerical value. Pedagogically, PSTs have an opportunity to analyze how high school students are using exponential properties and what assumptions they are implicitly drawing upon.

With these key points in mind, ask PSTs to solve the Williams Item. As they do so, ask them to think about the following questions:

- What is the high school student's logic? How might they have arrived at each step of their solution?
- What properties about exponentiation and arithmetic might they have used?
- Look at the strategies from the opener. Which one, if any, do Craig and Katlynn's work most resemble?

The PSTs might fill out a table like this for each high school student and then determine whether the high school student's work provides a mathematically valid application of exponential properties.

The steps of Craig's strategy are . . .	I think that Craig does NOT understand . . .	Craig evaluated $\left((-9)^{\frac{1}{2}}\right)^2$ to be . . .

The steps of Katlynn's strategy are . . .	The exponentiation and arithmetic properties that Katlynn used might be . . .	Katlynn evaluated $\left((-9)^{\frac{1}{2}}\right)^2$ to be . . .

Discussion for the Case of Ms. Williams's Class

Specific points to attend to for each high school student are the following:

- Craig's strategy resembles the second strategy in the opener (using power of a power property and commutative property). His strategy could be represented as follows:

$$\left((-9)^{\frac{1}{2}}\right)^2 = (-9)^{\frac{1}{2}\cdot 2} = (-9)^{2\cdot\frac{1}{2}} = ((-9)^2)^{\frac{1}{2}} = 81^{\frac{1}{2}} = 9.$$

- Craig also uses the convention that $x^{\frac{1}{2}}$ denotes only the positive square root of a number and not the negative square root.
- Katlynn's strategy resembles the first or the third strategy in the opener, depending on how "cancel" is interpreted. Her strategy could be represented as follows:

$$\left((-9)^{\frac{1}{2}}\right)^2 = (-9)^{\frac{1}{2}\cdot 2} = (-9)^1 = -9.$$

- Or, Katlynn might say that the square of a square root is always the original number, the square of a square root of -9 has to be -9.

Note that some PSTs may focus on evaluating the expression using complex numbers, which is a valid method for dealing with the problem, but it does not help us understand Craig's and Katlynn's methods, which are the focus. If this comes up, use the opportunity to discuss the difference between knowing a correct way to solve a problem yourself and knowing how to make sense of a method a student brings to the table.

At this point, PSTs might give an initial evaluation of whether Craig's and Katlynn's applications of exponential properties are valid. This leads to a central discussion question for this lesson: How is it possible that one expression can evaluate to two different values? PSTs may not be able to resolve this question. It is important to get PSTs to understand that there is something more here than meets the eye. One way to facilitate this discussion is to ask PSTs to compare and contrast the opener expression with the expression in Ms. Williams's class: What are similarities and differences? Some things to notice, to set up the final discussion for the lesson, are these:

- This expression and the one in the opener are very similar. Yet the opener expression always led to one answer, whereas the expression in Ms. Williams's class does not.
- The main difference is that the opener expression featured a positive base, and the expression in Ms. Williams's class featured a negative base.

There is another possible difference, which is that the opener base is not the size of a perfect square whereas the base in Ms. Williams's class is (because 9 is a perfect square). This is fine to notice, although this observation doesn't turn out to be a key difference. If PSTs notice this difference, record it and revisit its relevance after the next and final discussion.

Systematizing Our Observations: Domains and Exponentiation

The purpose of this section is to systematize our observations with exponential properties. There is something to be resolved: Why does it seem like sometimes exponential expressions have more than one value and other times they have exactly one value as we believe that numerical expressions should have? Is there a pattern to which expressions have exactly one value and which seem like they have more than one value?

To begin this section, ask PSTs to do the following:

- We are trying to figure out when exponential expressions have exactly one value and when it seems like they have more than one value. In other words: When are exponential expressions well-defined? Look over different strategies that Ms. Williams's students found and those from the opener.
- Use these strategies to find the values of $((-7)^2)^{\frac{1}{2}}$ and $((-7)^{-2})^3$. What do you notice? What questions do you have?

Evaluating the Expression $((-7)^2)^{\frac{1}{2}}$

With PSTs, unpack the strategies then summarize the punch line and elicit a conjecture about the guiding question: When do exponential expressions have exactly one value? When does it seem like they have more than one value?

Punch line: This expression might have three different values: −7, 7, and undefined. We might say the same about the expression in Ms. Williams's class.

Conjecture: Positive base expressions have one value. Expressions with negative bases seem to have different values.

Unpacking the Strategies

Strategy	Result	Notes
$((-7)^2)^{\frac{1}{2}} = (-7)^{2\cdot\frac{1}{2}}$ $= (-7)^1$ $= -7$	$((-7)^2)^{\frac{1}{2}} = -7$	
$((-7)^2)^{\frac{1}{2}} = (-7)^{2\cdot\frac{1}{2}}$ $= (-7)^{\frac{1}{2}\cdot 2}$ $= \left((-7)^{\frac{1}{2}}\right)^2 = \text{undefined}$	$((-7)^2)^{\frac{1}{2}}$ is undefined	$(-7)^{\frac{1}{2}}$ is undefined
$((-7)^2)^{\frac{1}{2}} = (-7)^{2\cdot\frac{1}{2}}$ $= (-7)^{\frac{1}{2}\cdot 2}$ $= \left((-7)^{\frac{1}{2}}\right)^2 = \left(\sqrt{7}i\right)^2$ $= \left(\sqrt{7}\right)^2 i^2$ $= -7$	$((-7)^2)^{\frac{1}{2}} = -7$	Uses power of a product $(xy)^a = x^a y^a$, and the definition of i as a square root of −1.
$((-7)^2)^{\frac{1}{2}} = (49)^{\frac{1}{2}} = 7$	$((-7)^2)^{\frac{1}{2}} = 7$	By definition of squaring and square root, and the convention that $x^{\frac{1}{2}}$ denotes the positive root.

Evaluating the Expression $((-7)^{-2})^3$

Unpack the strategies for PSTs as needed. There are a number of strategies that can be used, but they all result the same: The expression always evaluates to $1/7^6$. The main property used here is that x^{-a} is defined as $1/x^a$ (for nonzero *a*).

Revised conjecture: Positive base expressions have one value. A negative base can return one value when there are integer powers, even negative powers. But when negative bases are combined with fractional powers, the expression seems to result in different values.

Testing the conjectures: If there is time, PSTs might come up with other examples to test their conjecture. Does it matter what kind of fractions are used? Which order the fractions are used? What if there are two fractions?

Making Sense of Our Discoveries

To make sense of our findings, this lesson now turns to a review of how properties of exponents are defined.

Exponentiation is first defined for positive integer exponents, as repeated multiplication.

- $9^1 = 9, 9^2 = 9 \cdot 9 = 81, 9^3 = 9 \cdot 9 \cdot 9 = 729, \ldots$
- $x^1 = x, x^2 = x \cdot x, x^3 = x \cdot x \cdot x, \ldots$

We notice that exponentiation in these cases always satisfies the following properties, due to properties we know about multiplication:

- Power of a power property: $x^{ab} = (x^a)^b$.
- Power of a product property: $(xy)^a = x^a y^a$.
- Additive property of exponents: $x^a x^b = x^{a+b}$.

From here, we like these properties so much that we define other exponential expressions based on these properties. So the following properties are not provable in the sense that these properties did not have to necessarily hold. It's just that we, as mathematicians, wanted them to hold because they are nice and natural. So we think about these properties as conventions rather than as theorems:

- $x^0 = 1$.
- x^{-a} is defined as $1/x^a$ (when *a* is a nonzero integer).

But what about fractional exponents? This is when things get hairy.

- We want $x^{\frac{1}{n}}$ to be defined as the *n*th root of *x*.
- When *x* is positive, we already have a problem, at least for even *n*, because there might be more than one root.

- This is why we define $x^{\frac{1}{n}}$ to always be the positive root.
- We want expressions to only have one value.

When *x* is negative, how we treat this depends on the grade level.

- In middle school, we say that square roots are undefined for negative numbers, because there is no real number square root. Similarly, other even powered roots are undefined, but odd powered roots are defined (e.g., the cube root of -8 is -2.

When the quantity of *i* is introduced, we define the square root of -1 as *i*. But this raises the same problems as square roots with positive *x.* This is because both +*i* and $-i$ are square roots of -1. In the same way, all negative integers have two square roots.

- One way to resolve this issue is to define $-1^{\frac{1}{2}}$ and $\sqrt{-1}$ to be the positive power of *i*.
- We also then define $x^{\frac{1}{2}}$ and $\sqrt{x}$ to always be $\sqrt{|x|}\ i$, when *x* is negative.
- In advanced mathematics, another way to resolve the issues is to allow for one expression to have multiple values and to say that $-1^{\frac{1}{2}}$ is $\pm i$ and $x^{\frac{1}{2}}$ is $\pm\sqrt{|x|}\ i$.

Let's go back to problematic expressions and, instead of using exponential properties (since those seem to be giving us problems), interpret the expressions using strictly order of operations and the above definitions. Here are some expressions we would expect to be problematic:

- $\left((-9)^{\frac{1}{2}}\right)^2$, $\left((-7)^{\frac{1}{2}}\right)^2$, $((-9)^2)^{\frac{1}{2}}$, and $((-7)^2)^{\frac{1}{2}}$.
- Possibly others that the PSTs proposed.

If we use strictly order of operations and the above definitions, we must do the operations for the innermost parentheses grouping before the outside. Let's focus on the examples $\left((-9)^{\frac{1}{2}}\right)^2$ and $((-9)^2)^{\frac{1}{2}}$.

In middle school, the interpretation looks like this:

- The expression $\left((-9)^{\frac{1}{2}}\right)^2$ is undefined because the inside expression $(-9)^{\frac{1}{2}}$ is undefined.

- The expression $((-9)^2)^{\frac{1}{2}}$ can be defined, because the inside expression $(-9)^2 = 81$, and so $((-9)^2)^{\frac{1}{2}}$ equals 9.

In high school, after complex numbers are defined:

- The expression $\left((-9)^{\frac{1}{2}}\right)^2$ equals -9 because the inside expression $(-9)^{\frac{1}{2}}$ is defined as $3i$, and its square is -9. (Notice this also works for the advanced mathematics definition using $\pm 3i$).

- The expression $((-9)^2)^{\frac{1}{2}}$ is still defined to equal 9.

A way to generalize what we are finding is this:

- When working with negative bases, using exponential properties may lead us to contradictory values (the expression may be ill-defined) because we may be removing places where we use the convention of how $x^{\frac{1}{2}}$ is defined.

One way to think about this is that the domain of exponential properties is primarily for positive bases.

Working with order of operations and definitions always works, and this is how to determine the value of expressions no matter the base. With positive bases, this gives the same value as using exponential properties, and with negative bases, this may not because the exponential properties don't work over this domain when the powers are rational numbers that are not integers.

Responding to Craig and Katlynn

Recall that we interpreted Craig and Katlynn's thinking as follows:

- Craig's thinking: $\left((-9)^{\frac{1}{2}}\right)^2 = (-9)^{\frac{1}{2}\cdot 2} = (-9)^{2\cdot\frac{1}{2}} = ((-9)^2)^{\frac{1}{2}} = 81^{\frac{1}{2}} = 9$.

- Katlynn's thinking: $\left((-9)^{\frac{1}{2}}\right)^2 = (-9)^{\frac{1}{2}\cdot 2} = (-9)^1 = -9$

or

- Because the square of a square root is always the original number, the square of a square root of -9 has to be -9.

Ask PSTs to work in pairs on the following questions. As they talk, listen to their responses.

- Thinking about these generalizations, how would you respond to Craig and Katlynn? Are their applications of exponential properties valid? What do you think they understand? What might they not understand?

Points to summarize include the following:

- Craig's application is not mathematically valid. Katlynn's application, interpreted the first way, is not mathematically valid. Katlynn's application, interpreted the second way, is valid.
- When the applications are not mathematically valid, it is because the students are using exponential properties for a case in which they do not work: negative bases with fractional exponents. (Although Katlynn does evaluate the "right" answer, using the power of a power property does not work in general, as demonstrated by Craig's work. When a property can be used in ways that both work and do not work, we say that the property does not hold for those cases.)
- Katlynn's application is potentially valid if her reasoning were consistent with working with the expression using order of operation. This point is something that Ms. Williams would be wise to follow up with Katlynn about, as it is unclear from her statement how she was thinking about the "cancel."

Themes of the Discussion

The Williams minicase provided an opportunity for PSTs to practice attending to and making sense of student work. The examples and the opener also allowed PSTs to explore when an exponential expression is well-defined. This mathematical idea of "well-defined" is what drives the central inquiry.

Two other mathematical ideas that we discussed were conventions and extending domains of operation over time. We talked about how an expression might have the value of undefined before complex numbers are introduced and that it may have a numerical value after complex numbers are introduced. Although we do not always use the term undefined, the idea of being able to assign values to expressions only after more knowledge is not new.

For example, in elementary school, before negative numbers have been introduced, students generally do not "subtract bigger numbers from smaller numbers." Expressions such as $5-9$ might as well be undefined. But a question such as "Where is $5-9$ on the number line?" might be a way to begin talking about negative numbers. And after negative numbers have been introduced, we have a value for $5-9$, and it is -4. We can now also talk about how to add and subtract negative numbers. The domain of addition and subtraction has now been extended to all integers, where it was previously positive numbers. Similarly, there are certain quadratic equations whose roots "do not exist" prior to learning complex numbers. After learning complex numbers, we can identify complex roots and even do complex arithmetic. The domain of the square root operation is extended from positive reals and 0 to all real numbers.

In this lesson, we talked about how to extend the domain of exponentiation more generally to negative bases. We learned that to do so, we may not be able to use all the exponential properties we are accustomed to, but we can still go back to order of operation and definitions for exponentiation to evaluate expressions.

Hand in hand with the idea of extending domains is the notion of conventions. One convention from elementary school is that negative numbers are denoted with a minus sign. When we discovered negative numbers, we had to have a way of writing them. The convention for writing a square root of −1 is *i*. And when we were extending exponentiation from repeated multiplication to powers that were negative, 0, and powers of the form $\frac{1}{n}$ (for nonzero integer *n*), we used the convention that the properties of additive, power of a power, and power of a product should hold. This all worked for positive and 0 bases. Only when we extended the domain to negative bases did we have a problem and have to go back to original definitions including order of operations.

Closing

In general, when attending to and making sense of student work regarding algebraic properties, think about the following points:

- What is the high school student's logic? How might they have arrived at each step of their solution?
- What properties about exponentiation and arithmetic might they have used?
- When making sense of their strategies, attend to the properties and their domains.

We discovered the following:

- When working with negative bases, using exponential properties may lead us to contradictory values because we may be removing places where we use the convention of how $x^{\frac{1}{2}}$ is defined.
- One way to think about this is that the domain of exponential properties is primarily for positive bases.
- Working with order of operations and definitions always works, and this is how to determine the value of expressions no matter the base. However, positive bases give same value as using exponential properties, and negative bases may not because the exponential properties don't work.

Two themes that we talked about follow:

- We want to work with expressions in a way that their values are well-defined.
- In elementary school through high school and beyond, we see examples of extending domains of operation over time.

When domains are extended, and when we want expressions to be well-defined, we develop conventions.

Appendix B: Additional Discussion Prompts

This appendix provides teacher educators more ideas to prompt discussion with PSTs with regards to the Williams item in addition to the sample lesson plan for the Williams minicase.

- What might you do next instructionally to address the ideas Katlynn and Craig have put on the table? What are one or two mathematical understandings you would most want students to develop? How would your instructional moves help them to develop these understandings?
- One high school student could argue that the "right" answer to the problem of the Williams item, $\left((-9)^{\frac{1}{2}}\right)^2$, is -9, because this is the value that the expression would simplify to in the complex number system. How would you address a situation in which one high school student, who has seen complex numbers, argues that this is the case, and another high school student argues that it can't be because they aren't working with imaginary numbers? What are one or two mathematical understandings you would most want students to come to after participating in this disagreement? How would your instructional moves help them to develop these understandings?
- Imagine that a third high school student jumps into the conversation with Craig and Katlynn, saying:

  "In the textbook we have problems that are like simplify $(x^2)^{\frac{1}{2}}$, so how are we supposed to simplify anything with variables in it if there are different rules for negative and positive *x*'s?"

  How would you respond to the high school student?
- Imagine that another student attempts to graph $y = (-9)^{\mathrm{x}}$ using a graphing calculator and asks you to explain why it doesn't work. How would you respond? How does this response relate to the given problem of the Williams item?

- Consider the set of exponent "rules" that are often given to high school students for use in simplifying exponential expressions (listed below, but also feel free to include other variants). Which are taken as assumptions? Which can be proven from those assumptions? Which proofs depend on assumptions that the base is an integer and/or positive number?
    - $a^n \cdot a^m = a^{n+m}$
    - $(a^n)^m = a^{n \cdot m}$
    - $a^n = \frac{1}{a^{-n}}$
    - $a^0 = 1$
    - $a^{m/n} = \sqrt[n]{a^m}$
    - $a^{m/n} = \frac{1}{a^{-m/n}}$
- Consider the following examples of mathematical objects that may be left undefined earlier in a mathematical trajectory and later can become defined within a different number system.
    - negative numbers
    - square roots of negative number
    - division by zero (search the internet for "zero divisors")

Imagine in a conversation with a colleague that these examples were raised and your colleague has taken a stance that the right way to handle the first two instructionally is to signal to elementary and middle school students that they are "something coming later," but that the third example is something most K–12 students will never encounter in their mathematical trajectories and therefore not worthy of mention. What would your response be? How can you decide when later, more complicated mathematics is worth anticipating and when it is not?

Appendix C: Resources and References

This appendix provides additional resources that are relevant to the mathematics and/or teaching practices mentioned in this minicase. In particular, the two documents from the Education Development Center can be potentially assigned as reading for PSTs.

Abe, T., & Khosraviyani, F. (2009). Problems with rational exponents in elementary mathematics. *Texas College Mathematics Journal, 6*(1), 1–17.

Devlin, K. (2008). If it ain't repeated addition, what is it? [Blog post]. *Denise Gaskins' Let's Play Math*. https://denisegaskins.com/2008/07/01/if-it-aint-repeated-addition/

Education Development Center. (2016). *Rational exponents*. http://mathpractices.edc.org/pdf/Rational_Exponents.pdf

Education Development Center. (2016). *Extending patterns with exponents.* http://mathpractices.edc.org/pdf/Extending_Patterns_with_Exponents.pdf

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics*, *29*(1), 1–20.

Howe, R., & Epp, S. S. (2008). *Taking place value seriously: Arithmetic, estimation, and algebra*. Mathematics Association of America. https://www.maa.org/sites/default/files/pdf/pmet/resources/PVHoweEpp-Nov2008.pdf

Appendix D: Frequently Asked Questions

Where did the assessment items come from?

These items were produced by ETS staff in 2013 in an effort to determine how and how well item designs targeting elementary level MKT would extend to the secondary level, and the items were later utilized in a validity study (NSF grant number 1445630/1445551; https://www.nsf.gov/awardsearch/showAward?AWD_ID=1445630). The team that conducted the NSF work maintains an active Google Drive to provide access to items and elaborated answer rationale documents for the entire pool of items to interested scholars. If you are interested in joining this group, contact the ETS lead for the secondary MKT work, Heather Howell, hhowell@ets.org.

There's something I would like to change about the item. / I don't agree with the way the math is presented in the item. Would you consider changing it?

We decided in our work on the minicases to use the assessment items exactly as they were provided by the projects they came from. One goal of the further development work is to explore how existing intellectual capital in the form of assessments can be repurposed into material for teacher learning. The minicases have developed organically across a set of projects over a number of years, and there have been many contributors to them. The latest versions were reviewed by four experts in the field of mathematics and mathematics education, and their advice has been incorporated into revisions.

Part of what we want to illustrate is that the assessment item itself need not be above critique for it to be a useful starting point for PST learning. In fact, we think some critique might signal rich points for discussion as part of teacher development. That said, the point of the minicase is to be provocative, not prescriptive, and we encourage anyone who wishes to tweak, alter, subvert, delete, or completely rewrite the assessment item in service of their own instructional goals to do so. (And if it's an item from the Google Drive, we hope you'll post your work back in the drive for others to use!)

I would like to use these items as a hiring screen for new teachers, where could I find more of them?

This is not an approved use of these items. Accessing these items requires that you agree to terms of use that exclude high-stakes decision making.

Where could I find more minicases like these?

We only have a few exemplars ready for use at the current time, but we are more than happy to share them on request. To be added to our distribution list, contact Heather Howell, hhowell@ets.org.