Hierarchical Models for Social Networks

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Outline

1 Motivation

2 Estimation of Node/Dyad Attributes
   - Effect of Teaching the Same Grade with HLSM

3 Estimation of Network Attributes
   - Effect of Network Covariate with HMMSBM
Motivation

Social Networks in Education

Independent network replications

► Teacher networks within schools (Frank et al., 2004; Moolenaar et al., 2010; Spillane et al., 2012; Weinbaum et al., 2008)

► Student networks within classes/schools (Gest and Rodkin, 2011; Harris et al., 2008)
Motivation

Social Networks in Education

Independent network replications

- Teacher networks within schools (Frank et al., 2004; Moolenaar et al., 2010; Spillane et al., 2012; Weinbaum et al., 2008)
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How can we accommodate multiple networks using social network models?

- Modeling multiple networks simultaneously
- Estimating treatment effects
Hierarchical Network Models (Sweet et al., 2013, JEBS)
Hierarchical Network Models

\[ P(\mathbb{Y}|\mathbb{X},\Theta) = \prod_{k=1}^{K} P(Y_k|X_k = (X_{1k}, \ldots, X_{Pk}), \Theta_k = (\theta_{1k}, \ldots, \theta_{Qk})) \]

\[(\Theta_1, \ldots, \Theta_K) \sim F(\Theta_1, \ldots, \Theta_K|W_1, \ldots, W_K, \psi),\]

- \(P(Y_k|X_k, \Theta_k)\) is a model for a single network \(Y_k\) with covariates \(X_k\) and parameters \(\Theta_k\)
- \(W_k\) can model a variety of dependence assumptions across networks
- \(\psi\) may specify additional hierarchical structure on \(\Theta\)
Hierarchical Latent Space Model (HLSM)

Level 1

\[
\text{logit } P[Y_{ijk} = 1] = \beta_{0k} + \beta_{1k} X_{ijk} - |Z_{ik} - Z_{jk}|
\]

Level 2

\[
\beta_{0k} \sim \mathcal{N}(\mu_0, \sigma_0^2) \\
\beta_{1k} \sim \mathcal{N}(\mu_1, \sigma_1^2)
\]

- $X_{ijk}$ is a node-, tie-, or network- level covariate
- $Z_{ik}$ is the latent space position for actor $i$ in network $k$
An Example

Teacher Advice Network Data (Pitts and Spillane, 2009)

- 15 Elementary/K-8 schools
- 2D latent space positions
- Network size of 14 - 76 teachers
- Coded model fitting algorithm (MCMC) in R

Fitting the Model

Level 1

\[
\text{logit } P[Y_{ijk} = 1] = \beta_0 + \beta_1 X_{ijk} - |Z_{ik} - Z_{jk}|
\]

- \(Y_{ijk} = 1\) if teacher \(i\) asked \(j\) for advice
- \(X_{ijk} = 1\) if teachers \(i\) and \(j\) teach the same grade
  - \(\beta_1\) is the effect of teaching the same grade for school \(k\)
Level 1 Parameters $\beta_1$: Effect of Teaching the Same Grade

Separate Network Models

HLSM
Level 2 \((\mu_1, \sigma^2_1)\) Effect of Teaching the Same Grade

Fitting the Model

Level 1

\[
\text{logit } P[Y_{ijk} = 1] = \beta_{0k} + \beta_{1k}X_{ijk} - |Z_{ik} - Z_{jk}|
\]

Level 2

\[
\beta_{0k} \sim N(\mu_0, \sigma^2_0)
\]
\[
\beta_{1k} \sim N(\mu_1, \sigma^2_1)
\]

- \(\mu_1\) is the mean effect of teaching the same grade
- \(\sigma^2_1\) is the variance for the effect of teaching the same grade
Mixed Membership Stochastic Blockmodels

1. Models are for networks with subgroup structure
2. Each individual belongs to one of $n$ groups with some probability
3. Probability of a tie within subgroups is much higher than between subgroups

Typical blockmodels assume each individual belongs to one subgroup

Mixed membership allows individuals to belong to multiple subgroups
Mixed Membership in Networks

Network from Blockmodel  Network from MM Blockmodel
Mixed Membership Stochastic Blockmodel (Airoldi et al., 2008)

\[ Y_{ij} \sim Ber \left( S_{ij}^T B R_{ji} \right) \]
\[ S_{ij} \sim Multi \left( \theta_i \right) \]
\[ R_{ji} \sim Multi \left( \theta_j \right) \]
\[ \theta_i \sim Dir \left( \xi \gamma \right) \]
\[ B_{\ell m} \sim Beta \left( a_{\ell m}, b_{\ell m} \right) \]

- \( B_{sr} \) is the probability of a tie from group \( s \) to group \( r \)
- \( S_{ij} \) is the group membership of node \( i \) when sending a tie to person \( j \)
- \( R_{ji} \) is the group membership of node \( j \) when receiving a tie from person \( i \)
- \( \theta_i \) is the membership probability vector for person \( i \)

Small values of \( \gamma \) generate extreme the membership probabilities
\( \gamma : \) the Amount of Mixed Membership

Figure: \( \xi = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}, \gamma = 0.09 \)

Figure: \( \xi = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}, \gamma = 0.60 \)
Is the Amount of Mixing Related to Network Attributes?

Model Multiple Networks Simultaneously
- Higher efficiency in educational research
- Make multiple networks comparable

Figure: Selected 5th Grade Cognitive Friendship Networks
The Hierarchical Mixed Membership Stochastic Blockmodel with Network-level Covariates

\[ Y_{ijk} \sim \text{Ber}\left(S_{ijk}^T B_k R_{jik}\right) \]
\[ S_{ijk} \sim \text{Multi}\left(\theta_{ik}\right) \]
\[ R_{jik} \sim \text{Multi}\left(\theta_{jk}\right) \]
\[ \theta_{ik} \sim \text{Dir}\left(\xi_k \gamma_k\right) \]
\[ \gamma_k = \exp\left(\beta' X_k\right) \]
\[ B_{\ell mk} \sim \text{Beta}\left(a_{\ell mk}, b_{\ell mk}\right) \]
\[ \xi_k \sim \text{Dir}\left(1\right) \]
\[ \beta \sim \text{MVN}\left(\mu, \sigma^2\right) \]

- \( X = (X_1, \ldots, X_K) \) is a set of network-level covariates
Real Data Analysis with 5th Grade Friendship Networks (Gest and Rodkin, 2011)

- Data:
  - Cognitive friendship adjacency matrices in 5th grade, 17 classrooms
  - Average classroom size (network nodes): 20
  - Teacher Management

- Hypothesis: Teachers with higher social management have friendship networks with less isolated subgroups

- Model: 4-group HMMSBM with Network-level Covariates
Posterior Density: $\beta$ Effect of Cognitive Facilitation

Significantly positive $\beta$: Teachers with higher Cognitive Facilitation have friendship networks with higher probability of mixing
Positive Relationship between Teacher Management and Mixed Membership

\[ CF = -0.56 \]

\[ CF = 0.03 \]

\[ CF = 0.75 \]


For More Information:
http://hnm.stat.cmu.edu or tsweet@umd.edu

Current Work
- HLSMs for covariate effects and interventions
- Power Analysis for HLSMs
- Blockmodels and Mixed Membership
- Other ways to incorporate covariates into blockmodels
- HNMs for Mediation and Influence

Future Work
- Develop new models and extensions (longitudinal models)
- Operating characteristics
- Identifiability and estimation issues
- Applications with educational data
- Valued ties
HLSM for Influence

\[
\begin{align*}
\text{logit } P[Y_{ij} = 1] &= \beta^t X_{ij} - d(Z_i, Z_j) \\
W_i &= \phi_0 W_i^* + \phi N(Z)_i W_i^* + \nu + \epsilon_i
\end{align*}
\]

- $W_i$ is the covariate for node $i$ at time 2
- $\phi$ denotes the effect of the network
- $W_i^*$ is the covariate for node $i$ at time 1
- $N(Z)_i$ is the weight of the network distances
Extra Slides
Fix Group Number in the Model across Networks

- 24 nodes in each network.
- True $\gamma$ values are the same for all networks.
- Fit all networks with 4-group MMSBM.

$\hat{\gamma} = 0.07$  
$\xrightarrow{\text{increased by 29\%}} \hat{\gamma} = 0.09$  
$\xrightarrow{\text{increased by 44\%}} \hat{\gamma} = 0.13$
Fitting the HLSM for Interventions

Fitted Model

\[ \text{logit} \ P[Y_{ijk} = 1] = \beta_0 + \beta_{1k}X_{ijk} - |Z_{ik} - Z_{jk}| + \alpha T_k \]

\[ Z_{ik} \sim MVN \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \right) \]

\[ \beta_0 \sim N(0, 100) \]

\[ \beta_{1k} \sim N(0, 100) \]

\[ \alpha \sim N(0, 100) \]

\( X_{ijk} \) is the indicator that teacher \( i \) and \( j \) in network \( k \) teach the same grade

Coded model fitting algorithm (MCMC) in R
Fitting a HLSM

\[
\text{logit } P[Y_{ijk} = 1] = \beta_{0k} + \beta_{1k}X_{1ijk} - |Z_{ik} - Z_{jk}|,
\]

\[
Z_{ik} \sim \text{MVN} \begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix}, \quad i = 1, \ldots, n_k,
\]

\[
\beta_{0k} \sim N(\mu_0, \sigma_0^2), \quad k = 1, \ldots, K,
\]

\[
\beta_{1k} \sim N(\mu_1, \sigma_1^2), \quad k = 1, \ldots, K,
\]

\[
\mu_0 \sim N(b_1, b_2),
\]

\[
\mu_1 \sim N(c_1, c_2),
\]

\[
\sigma_0^2 \sim \text{Inv - Gamma}(d_1, d_2),
\]

\[
\sigma_1^2 \sim \text{Inv - Gamma}(e_1, e_2),
\]

\(X_{ijk}\) is the indicator that teacher \(i\) and \(j\) in network \(k\) teach the same grade
Mixed Membership Stochastic Blockmodels

1. Models are for networks with subgroup structure
2. Each individual belongs to one of \( n \) groups with some probability
3. Probability of a tie within subgroups is much higher than between subgroups

Typical blockmodels assume each individual belongs to one subgroup

Mixed membership allows individuals to belong to multiple subgroups
The Hierarchical Mixed Membership Stochastic Blockmodel (HMMSBM)

\[ Y_{ijk} \sim \text{Bernoulli}(S_{ijk}^T B_k R_{jik}) \]
\[ S_{ijk} \sim \text{Multinomial}(\theta_{ik}, 1) \]
\[ R_{jik} \sim \text{Multinomial}(\theta_{jk}, 1) \]
\[ \theta_{ik} \sim \text{Dirichlet}(\lambda_k) \]
\[ B_{\ell mk} \sim \text{Beta}(a_{\ell mk}, b_{\ell mk}) \]

- \( B_k[S, R] \) is the probability of a tie from group S to group R in network k
- \( S_{ijk} \) is the group membership of person i when sending a tie to person j
- \( R_{jik} \) is the group membership of person j when receiving a tie from person i
- \( \theta_{isk} \) is the probability that person i in network k belongs to group s

Small values of \( \lambda_k \) generate extreme the membership probabilities
A HMMSBM for an Intervention

Suppose an intervention is hypothesized to change isolated subgroups...

\[ P(Y_{ijk} = 1) = B_k[S_{ijk}, R_{jik}] \]
\[ P(S_{ijk} = s) = \theta_{isk} \]
\[ P(R_{jik} = r) = \theta_{jrk} \]

\[ \theta_{ik} \sim Dirichlet(\lambda_k) \]
\[ \lambda_k = \lambda_0 + T_k(\vec{I} - \lambda_0)(1 - \alpha) \]

- \( T_k \) is treatment indicator
- \( \vec{I} \) is \((1, ... 1)\) with length equal to number of groups
- \( \alpha \) is the treatment effect
- \( \lambda_0 \) generates the control group’s membership probabilities
Simulated Data from 20 Networks of Size 20; \( \alpha = 0.53 \)

\[
\theta_{ik} \sim \text{Dirichlet}(\lambda_k)
\]

\[
\lambda_k = \lambda_0 + T_k(\bar{1} - \lambda_0)(1 - \alpha)
\]

\[
\lambda_k \rightarrow 0.05 + 0.45 T_k
\]
A HMMSBM for an Intervention

**Fitted Model**

\[
Y_{ijk} \sim Bernoulli(B_k[S_{ijk}, R_{ijk}])
\]

\[
S_{ijk} \sim Multinomial(\theta_{ik}, 1)
\]

\[
R_{jik} \sim Multinomial(\theta_{jk}, 1)
\]

\[
\theta_{ik} \sim Dirichlet(\lambda_k)
\]

\[
B_{\ell k} \sim Beta(3, 1)
\]

\[
B_{\ell m k} \sim Beta(1, 10), \ell \neq m
\]

\[
\lambda_k = \lambda_0 + T_k(\overrightarrow{1} - \lambda_0)(1 - \alpha)
\]

\[
\alpha \sim Uniform(0, 1)
\]

- Model fit using MCMC algorithm coded in R
- \( \lambda_0 = 0.05 \)
- 4 subgroups assumed *a priori*
Treatment Effect Parameter ($\alpha$) Recovery
Tie Probability Recovery